Indian Statistical Institute, Bangalore
M. Math. Second Year, Second Semester
Advanced Functional Analysis

Mid-terms Examination Date: 04-03-2015
Maximum marks: 100 Time: 3 hours

(1) Let X be a topological vector space over the field $\mathbb{C}$. Suppose $f : X \to \mathbb{C}$ is a linear functional. Show that f is continuous if and only if $\ker(f) = \{x \in X : f(x) = 0\}$ is closed in X. [10]

(2) Suppose K is a compact subset of a topological vector space Z. Show that $K - K$ is compact. [10]

(3) Let X be a separable Banach space with countable dense set $\{x_n : n \geq 0\}$. For $n \geq 1$, let f_n be a linear functional on X satisfying $f_n(x_n) = \|x_n\|$, and $\|f_n\| = 1$. Define $T : X \to l^\infty$ by $T(x) = (f_1(x), f_2(x), \ldots,)$. Show that T is an isometry. [This proves that every separable Banach space is isometrically isomorphic to a subspace of l^∞.] [20]

(4) Let X, Y be normed linear spaces and let X be finite dimensional. Suppose $T : X \to Y$ is a linear map. Show that T is continuous. [20]

(5) Let X, Y be Banach spaces. Let $T : X \to Y$ be a bounded linear map and T is onto. Show that T is an open map. [20]

(6) Let V be a topological vector space. A non-empty subset A of V is said to be *absorbing* if for each $x \in V$, there exists $t > 0$ such that $\frac{x}{t}$ is in A. Show that every open neighborhood of 0 is absorbing. Show that every $y \neq 0$ has an open neighborhood which is not absorbing. [20]

(7) Let $C[0, 1]$ be the Banach space of complex valued continuous functions on the interval, with supremum norm. Let $C_0[0, 1]$ be the subspace:

$$C_0[0, 1] = \{f \in C[0, 1] : f(0) = 0\}.$$

Identify the set of extreme points of closed unit balls of $C[0, 1]$ and $C_0[0, 1]$. [10]